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Melnikov Method for Homoclinic Bifurcation in Nonlinear Impact Oscillators

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Abstract—Based on an inverted pendulum impacting on rigid walls under external periodic excitation, a class of nonlinear impact oscillators is discussed for its homoclinic bifurcation The Melnikov method established for smooth dynamical systems is extended to be applicable to the nonsmooth one For nonlinear impact systems, closed form solutions between impacts are generally unavailable. The absence of closed form solutions makes difficulties in estimation of the gap between the stable manifold and unstable manifold In this paper, we give a method to compute the Melnikov functions up to the n^{th}-order so as to obtain conditions of parameters for the persistence of homoclinic cycles which are formed via the identification given by the impact rule. © 2005 Elsevier Ltd All rights reserved

Keywords—Impact oscillator, Nonsmooth system, Melnikov method, Homoclinic bifurcation, Poincaré section

1. INTRODUCTION

In recent years, there are growing interests in applied science and engineering on *piecewise smooth* (PWS) dynamical systems. These systems arise in many practical situations such as switching circuits in power electronics [1], vibro-impacting machines in mechanical engineering [2], walking machines [3], etc. The phase space of such systems can be split into one or more connected regions by the so-called *switching manifolds* and the vector fields are not smooth across the switching manifolds. Those systems often exhibit very complicated dynamics due to the nonsmoothness, including traditional bifurcations (e.g , period doubling and saddle-node bifurcations) as possessed by smooth vector fields and nonclassical bifurcations such as grazing, border-collision, and sliding effects, etc.

A class of the most studied PWS dynamical systems is impact oscillators where one or more walls are present serving as motion limiters. Typically, instantaneous impact with some energy loss is assumed during the contact of the oscillator with the wall. This subject has been dealt with in many papers, for example, [2,4-9]. In particular, homoclinic bifurcations and chaos were

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discussed in [5,9] for piecewise linear (PWL) systems with impacts like the form,

$$\begin{aligned} \ddot{x} + 2\alpha\dot{x} - x = \beta\cos\omega t, &\qquad \text{as } |x| < 1, \\ x \mapsto -r\dot{x}, &\qquad \text{as } |x| = 1. \end{aligned} \tag{1}$$

For system (1), the dynamics between switching manifolds $x = \pm 1$ is determined by a linear equation, from which closed form solutions between the switching manifolds can be obtained so that the contact points of an orbit to the impacting walls can be calculated precisely. Although the known works [5,9] give some good ideas in dealing with homoclinic bifurcation for impact oscillators, a general method is still needed for complicated nonlinear impact systems whose closed form solutions are not available. The lack of closed form solutions makes difficulties in estimation of the gap between the stable manifold and unstable manifold. In fact, many practical models demand general approach to the bifurcation problems in PWS dynamical systems including impact oscillators. See, e.g., [10,11] for more on these issues.

In this paper, we consider a general form of nonlinear impact oscillators, which can be used to model an inverted pendulum impacting on rigid walls under external periodic excitation as depicted in Figure 1. We can scale the gap size between the rigid walls to be two. The motion of the oscillator between the walls is governed by the differential equation,

$$\ddot{x} + g(x) = \varepsilon f(t, x, \dot{x}, \varepsilon), \qquad |x| < 1. \tag{2}$$

The impact law is given by

$$\dot{x} \mapsto -(1 - \varepsilon\rho)\,\dot{x}, \qquad |x| = 1, \tag{3}$$

where $|\varepsilon| \le \varepsilon_0 \ll 1$, for some $\varepsilon_0 > 0$ and $1 - \varepsilon\rho \in (0, 1]$ is the coefficient of restitution representing energy loss during impact. Assume that functions f and g satisfy the following hypotheses.

(H1) $g: J := (-\mu, \mu) \to \mathbb{R}$ is C^{n+1} ($n \ge 1$), where $\mu > 1$ and $g(0) = 0$, $g'(0) < 0$, $g(x) \ne 0$, for $x \in [-1, 0) \cup (0, 1]$.

(H2) $f: \mathbb{R} \times J \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ is C^{n+1} and T-periodic in t, such that εf has the following expansion in ε,

$$\varepsilon f(t, x, y, \varepsilon) = \sum_{k=1}^{n} f_k(t, x, y)\,\varepsilon^k + O\left(\varepsilon^{n+1}\right), \tag{4}$$

where each $f_k: \mathbb{R} \times J \times \mathbb{R} \to \mathbb{R}$ is C^{n+1} and T-periodic in t.

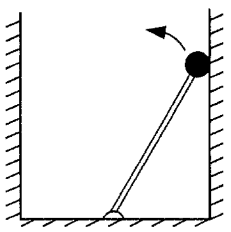

Figure 1 Inverted pendulum

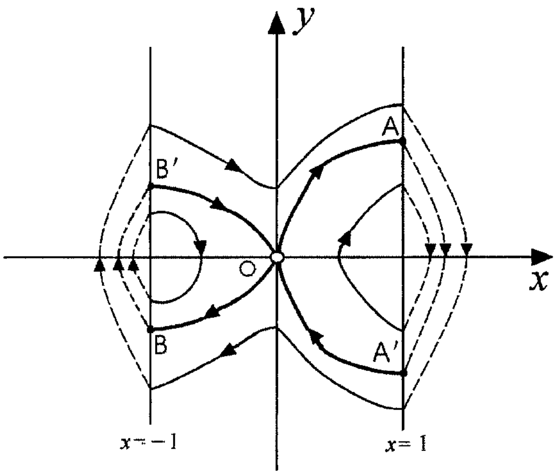

Figure 2 Orbits of unperturbed system.

For $\varepsilon = 0$, the unperturbed system describes a free impact oscillator and is equivalent to

$$\dot{x} = y, \quad \dot{y} = -g(x), \qquad |x| < 1, \tag{5}$$

$$y \mapsto -y, \qquad |x| = 1. \tag{6}$$

By Hypothesis (H1), system (5) has a unique equilibrium at the origin $O(0,0)$ for $x \in [-1,1]$, which is a saddle. Considering the identification given by impact rule (6), it is easy to see that the phase portrait of system (5),(6) is qualitatively the same as shown in Figure 2. Note that the unperturbed system (5),(6) has a saddle at the origin O and two homoclinic loops $\Gamma_+ = OAA'O$ and $\Gamma_- = OBB'O$ Here, A', B' are the reflection points of A, B, respectively. Usually when the excitation is added, these homoclinic orbits are destroyed. The question is under what conditions the stable manifold W^s and unstable manifold W^u intersect transversely. Such transversal intersections imply the existence of Smale horseshoes via the Smale-Birkhoff homoclinic theorem (see [12, p. 252]), which result in the appearance of chaotic motions.

In this paper, we extend results for PWL systems obtained by Chow and Shaw [5] and also by Shaw and Rand [9] to the general nonlinear impact system (2),(3). We present a general method of Melnikov type to determine whether transversal homoclinic intersection between its stable manifold and unstable manifold occurs under appropriate damping and external periodic excitation. We give a procedure for the computation of Melnikov functions up to the n^{th}-order.

Our paper is organized as follows. In Section 2, we introduce the Poincaré section and the Poincaré map for system (2),(3) and describe the separation between W^s and W^u in the Poincaré section. In Section 3, we describe our idea of calculating Melnikov functions and give formula for the first-order Melnikov function. In Section 4, we present a procedure to calculate higher-order Melnikov functions. Then, we apply our method to a PWL system and to a nonlinear PWS system in Section 5.

2. SEPARATION ON THE POINCARÉ SECTION

System (2),(3) can be rewritten in the form,

$$\dot{x} = y, \quad \dot{y} = -g(x) + \varepsilon f(t, x, y, \varepsilon), \qquad \text{as } |x| < 1, \tag{7}$$

$$y \mapsto -(1 - \varepsilon\rho)\, y, \qquad \text{as } |x| = 1. \tag{8}$$

Ignoring the walls $x = \pm 1$, system (7) is equivalent to the following suspended system,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(x) + \varepsilon f(\theta, x, y, \varepsilon), \\ \dot{\theta} &= 1, \end{aligned} \tag{9}$$

where $\theta = t (\operatorname{mod} T)$ and $x \in J$. By Lemma 4.5 1 in [12], for arbitrary $t_0 \in [0,T]$ and sufficiently small ε, system (9) has a unique hyperbolic periodic orbit $\gamma_\varepsilon^0(t) = O + O(\varepsilon) \in \mathbb{R}^2$, which has a stable manifold W_ε^s and an unstable manifold W_ε^u.

Because of the nature of the vector field (7),(8), the Poincaré section is taken to be the cylinder,

$$\Sigma := \{(\theta, x, y) \in S^1 \times I \times \mathbb{R} \mid x = 1,\ y > 0\} = S^1 \times \mathbb{R}^+,$$

where S^1 is the circle of period T. Let

$$\Sigma_- := \{(\theta, x, y) \in S^1 \times I \times \mathbb{R} \mid x = 1,\ y < 0\} = S^1 \times \mathbb{R}^-.$$

Then, for any $\theta \in [0,T]$ and sufficiently small ε, a branch of the unstable manifold W_ε^u of $\gamma_\varepsilon^0(t)$ intersects Σ in a curve, denoted by $\Delta_\varepsilon^u(\theta)$. Similarly, for any $\theta \in [0,T]$ and sufficiently small ε, a branch of the stable manifold W_ε^s intersects Σ_- in a curve, denoted by $\Delta_\varepsilon^s(\theta)$. The impact rule (8) defines a 1-1 map $\mathcal{I}:\Sigma\to\Sigma_-$, such that

$$(\theta,1,y)\mapsto(\theta,1,-(1-\varepsilon\rho)\,y)\,.$$

Hence, each point $\Delta_\varepsilon^s(\theta)$ on the curve Δ_ε^s is the image of the map $\mathcal{I}$ of the point $-(1-\varepsilon\rho)^{-1}\Delta_\varepsilon^s(\theta)$ on Σ. For each $\theta\in[0,T]$, let $\hat{\Delta}_\varepsilon^s(\theta):=\mathcal{I}^{-1}(\Delta_\varepsilon^s(\theta))\in\Sigma$. Then, the curve $\hat{\Delta}_\varepsilon^s$ is the preimage of Δ_ε^s under $\mathcal{I}$. The problem is when the two curves $\hat{\Delta}_\varepsilon^s$ and Δ_ε^u in Σ intersect transversely. This is equivalent to studying the following function, which we call the *separation* between W_ε^s and W_ε^u in Σ,

$$\Delta_\varepsilon(\theta):=\Delta_\varepsilon^u(\theta)+\frac{1}{1-\varepsilon\rho}\Delta_\varepsilon^s(\theta)\,,\qquad\theta\in[0,T]\,. \tag{10}$$

If $\Delta_\varepsilon(\theta)$ has no zeros, then W_ε^s and W_ε^u do not intersect, while if $\Delta_\varepsilon(\theta)$ has simple zeros, then W_ε^s and W_ε^u intersect transversally. If $\Delta_\varepsilon(\theta)$ has quadratic zeros, then W_ε^s and W_ε^u intersect with quadratic tangencies. By the Smale-Birkhoff homoclinic theorem (see [12, p. 252]), the transversal intersections of W_ε^s and W_ε^u imply the existence of Smale horseshoes, which result in the appearance of chaotic motions.

Generally, it is impossible to find a closed form of the function $\Delta_\varepsilon(\theta)$. The main purpose of this paper is to approximate $\Delta_\varepsilon(\theta)$ to the n^{th}-order. It is well known that (see [13, p. 489]) W_ε^s and W_ε^u are C^{n+1} in ε under Hypotheses (H1) and (H2). Thus, $\Delta_\varepsilon^u(\theta)$, $\Delta_\varepsilon^s(\theta)$ and $\Delta_\varepsilon(\theta)$ are all C^{n+1} in ε from the way they are defined. Suppose that $\Delta_\varepsilon^u(\theta)$ and $\Delta_\varepsilon^s(\theta)$ have the following Taylor expansions near $\varepsilon=0$,

$$\Delta_\varepsilon^u(\theta)=M_0^u(\theta)+\sum_{k=1}^{n}M_k^u(\theta)\,\varepsilon^k+O\left(\varepsilon^{n+1}\right), \tag{11}$$

$$\Delta_\varepsilon^s(\theta)=M_0^s(\theta)+\sum_{k=1}^{n}M_k^s(\theta)\,\varepsilon^k+O\left(\varepsilon^{n+1}\right). \tag{12}$$

Note that

$$\frac{1}{1-\varepsilon\rho}=1+\sum_{k=1}^{n}\rho^k\varepsilon^k+O\left(\varepsilon^{n+1}\right). \tag{13}$$

Substituting (11)–(13) into (10) yields

$$\Delta_\varepsilon(\theta)=M_0(\theta)+\sum_{k=1}^{n}M_k(\theta)\,\varepsilon^k+O\left(\varepsilon^{n+1}\right), \tag{14}$$

where

$$M_0(\theta):=M_0^u(\theta)+M_0^s(\theta),\qquad M_k(\theta):=M_k^u(\theta)+\sum_{m=0}^{k}\rho^m M_{k-m}^s(\theta). \tag{15}$$

Clearly, $M_0(\theta)$ is the separation between W_0^s and W_0^u in Σ, for $\varepsilon=0$ and $M_0^u(\theta)\equiv\sqrt{-2G(1)}$, $M_0^s(\theta)\equiv-\sqrt{-2G(1)}$, where

$$G(x):=\int_0^x g(s)\,ds,\qquad x\in J. \tag{16}$$

Hence, $M_0(\theta)\equiv 0$ by our assumption and therefore, (14) becomes

$$\Delta_\varepsilon(\theta)=M_1(\theta)\,\varepsilon+M_2(\theta)\,\varepsilon^2+\cdots+M_n(\theta)\,\varepsilon^n+O\left(\varepsilon^{n+1}\right), \tag{17}$$

where $\theta\in[0,T]$ and $|\varepsilon|\ll 1$. Thus, we obtain the following result.

THEOREM 2.1. *If $M_1(\theta_0)=0$ and $M_1'(\theta_0)\neq 0$, for some $\theta_0\in[0,T]$ or, more generally, if there is an integer k with $1\le k\le n$ and a number $\theta_0\in[0,T]$, such that $M_1(\theta)\equiv\cdots\equiv M_{k-1}(\theta)\equiv 0$, $M_k(\theta_0)=0$ and $M_k'(\theta_0)\neq 0$, then for sufficiently small $\varepsilon>0$, the manifolds W_ε^s and W_ε^u intersect transversally in Σ*

3. THE FIRST-ORDER MELNIKOV FUNCTION

Among all Melnikov functions $M_k(\theta)$, the first-order one, $M_1(\theta)$, is the most fundamental and useful. In this section, we prove a formula for its calculation in terms of given functions.

In general, the calculation of $M_k(\theta)$ for $1 \le k \le n$ is proceeded by computing $M_k^u(\theta)$ and $M_k^s(\theta)$ as shown in (15). Note that $\Delta_\varepsilon^u(\theta)$ and $\Delta_\varepsilon^s(\theta)$ are both the y-coordinates of the suspended autonomous orbits of (9) lying in W_ε^u and W_ε^s, respectively, at $x = 1$ and the fixed time $t = \theta$. Unlike the PWL systems discussed in [5,9], it is generally impossible to find a closed form solution of (9). So, it is difficult to determine the time for a given orbit of (15) at which it reaches the wall $x = 1$.

Our idea to overcome this difficulty is as follows. Select

$$P_0^u \cdot (x_0^u(0), y_0^u(0)) \qquad \text{and} \qquad P_0^s : (x_0^s(0), y_0^s(0))$$

on the unperturbed (i.e., $\varepsilon = 0$) unstable manifold W_0^u and stable manifold W_0^s of the saddle O of (9), respectively, but outside the wall $x = 1$, i.e.,

$$1 < x_0^u(0) < \mu, \qquad 1 < x_0^s(0) < \mu. \tag{18}$$

Let $\gamma_0^u : (x_0^u(t), y_0^u(t))$ and $\gamma_0^s : (x_0^s(t), y_0^s(t))$ denote the orbits of (9) for $\varepsilon = 0$ passing through P_0^u and P_0^s at $t = 0$ respectively, which of course lie in W_0^u and W_0^s, respectively. Let τ_0^u and τ_0^s be the time at which the orbits γ_0^u, γ_0^s reach the wall $x = 1$ respectively, i.e., $x_0^s(\tau_0^s) = x_0^u(\tau_0^u) = 1$. Clearly,

$$\tau_0^u < 0, \qquad \tau_0^s > 0. \tag{19}$$

In order to discuss W_ε^u and W_ε^s for perturbed system (9), we consider orbits of (8), for small $\varepsilon \ne 0$ with initial points on the normal line L^u of W_0^u at P_0^u and the normal line L^s of W_0^s at P_0^s, i.e., for arbitrarily chosen $t_0 \in \mathbb{R}$, let $\gamma_\varepsilon^u : (x_\varepsilon^u(t,t_0),\ y_\varepsilon^u(t,t_0))$ and $\gamma_\varepsilon^s : (x_\varepsilon^s(t,t_0),\ y_\varepsilon^s(t,t_0))$ be orbits of the perturbed system (9) lying in W_ε^u and W_ε^s, such that the vectors $(x_\varepsilon^u(t_0,t_0) - x_0^u(0),\ y_\varepsilon^u(t_0,t_0) - y_0^u(0))$ and $(x_\varepsilon^s(t_0,t_0) - x_0^s(0),\ y_\varepsilon^s(t_0,t_0) - y_0^s(0))$ are orthogonal to the unperturbed vector field of (9) at P_0^u and P_0^s, respectively. See Figure 3. By the continuous dependency on parameters and initial values, we have

$$1 < x_\varepsilon^u(t_0,t_0) < \mu, \qquad 1 < x_\varepsilon^s(t_0,t_0) < \mu,$$

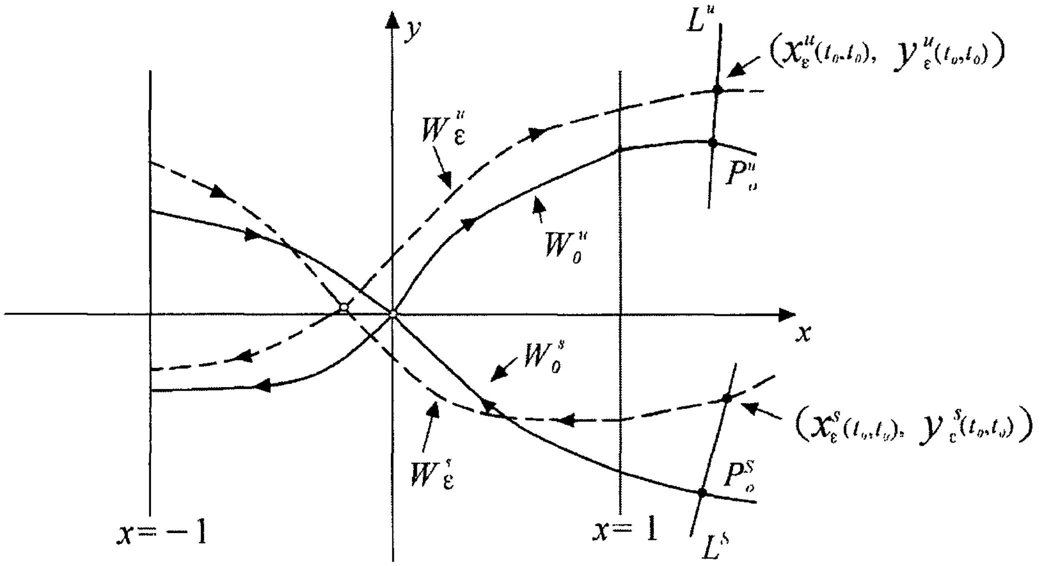

Figure 3 Perturbed invariant manifolds.

for sufficiently small ε. Then, we estimate the time $T^u(\varepsilon, t_0)$ and $T^s(\varepsilon, t_0)$ at which the orbits γ_ε^u and γ_ε^s reach the wall $x = 1$, so that γ_ε^u and γ_ε^s intersect the plane $x = 1$ in the phase space of the suspended autonomous system (9) at $(t_0, 1, y_\varepsilon^u(T^u(\varepsilon, t_0), t_0))$ and $(t_0, 1, y_\varepsilon^s(T^s(\varepsilon, t_0), t_0))$, respectively. Let

$$\tilde{\Delta}_\varepsilon^u(t_0) := y_\varepsilon^u(T^u(\varepsilon, t_0), t_0), \qquad \tilde{\Delta}_\varepsilon^s(t_0) := y_\varepsilon^s(T^s(\varepsilon, t_0), t_0). \tag{20}$$

Clearly, $\tilde{\Delta}_\varepsilon^u(t_0)$ and $\tilde{\Delta}_\varepsilon^s(t_0)$ are both the y-coordinates of the suspended autonomous trajectories of (9) lying in W_ε^u and W_ε^s, respectively, at $x = 1$ and at the times $t = T^u(\varepsilon, t_0)$ and $t = T^s(\varepsilon, t_0)$, respectively. Now, for any fixed $\theta \in [0, T]$, if we can solve the equations,

$$\theta = T^u(\varepsilon, t_0), \tag{21}$$

$$\theta = T^s(\varepsilon, t_0), \tag{22}$$

for t_0 and let $t_0^u(\varepsilon, \theta)$ and $t_0^s(\varepsilon, \theta)$ be the solutions of (21) and (22), respectively, then we have

$$\Delta_\varepsilon^u(\theta) = \tilde{\Delta}_\varepsilon^u(t_0^u(\varepsilon, \theta)), \qquad \Delta_\varepsilon^s(\theta) = \tilde{\Delta}_\varepsilon^s(t_0^s(\varepsilon, \theta)). \tag{23}$$

LEMMA 3.1. *All of $T^u(\varepsilon, t_0)$, $T^s(\varepsilon, t_0)$, $t_0^u(\varepsilon, \theta)$, and $t_0^s(\varepsilon, \theta)$ described above exist and are C^{n+1} in ε for sufficiently small ε.*

PROOF. We only prove the lemma for $T^u(\varepsilon, t_0)$ and $t_0^u(\varepsilon, \theta)$. The proof for $T^s(\varepsilon, t_0)$ and $t_0^s(\varepsilon, \theta)$ is similar.

By standard theory [14, p. 22], when the initial point moves smoothly along L^u starting from P_0^u at the initial time t_0 and the parameter ε varies, the solution of (9) is C^{n+1} smooth in both ε and the initial data under Hypotheses (H1) and (H2). Here, the solution of (9) under consideration for $\varepsilon = 0$ is $(x_0^u(t - t_0), y_0^u(t - t_0))$. In particular, $(x_\varepsilon^u(t, t_0), y_\varepsilon^u(t, t_0))$ is C^{n+1} in ε. It is easy to see that

$$\left.\frac{\partial x_\varepsilon^u(t, t_0)}{\partial t}\right|_{\varepsilon=0,\ t=t_0+\tau_0^u} = y_0^u(\tau_0^u) \neq 0.$$

By implicit function theorem (see Theorem 2.3 in [15, p. 26]), the solution $T^u(\varepsilon, t_0)$ of the equation, $x_\varepsilon^u(T^u(\varepsilon, t_0), t_0) = 1$, exists and is C^{n+1} in ε for sufficiently small ε.

By the same argument we also see that $T^u(\varepsilon, t_0)$ is C^{n+1} in (ε, t_0) for sufficiently small ε and $t_0 \in \mathbb{R}$. Take the derivative with respect to t_0 at $\varepsilon = 0$ on both sides of $x_\varepsilon^u(T^u(\varepsilon, t_0), t_0) = 1$ and note that $T^u(0, t_0) = t_0 + \tau_0^u$, we have

$$\begin{aligned}
0 &= \left.\frac{\partial x_\varepsilon^u(t, t_0)}{\partial t}\right|_{\varepsilon=0,\ t=T^u(0,t_0)} \cdot \left.\frac{\partial T^u(\varepsilon, t_0)}{\partial t_0}\right|_{\varepsilon=0} + \left.\frac{\partial x_\varepsilon^u(t, t_0)}{\partial t_0}\right|_{\varepsilon=0,\ t=T^u(0,t_0)} \\
&= y_0^u(\tau_0^u)\ \left.\frac{\partial T^u(\varepsilon, t_0)}{\partial t_0}\right|_{\varepsilon=0} + \left.\frac{\partial x_0^u(t - t_0)}{\partial t_0}\right|_{t=T^u(0,t_0)} \\
&= y_0^u(\tau_0^u) \cdot \left.\frac{\partial T^u(\varepsilon, t_0)}{\partial t_0}\right|_{\varepsilon=0} - y_0^u(\tau_0^u).
\end{aligned}$$

Thus, for any $t_0 \in \mathbb{R}$,

$$\left.\frac{\partial T^u(\varepsilon, t_0)}{\partial t_0}\right|_{\varepsilon=0} \equiv 1.$$

Again, by implicit function theorem, for any $\theta \in [0, T]$, the solution $t_0^u(\varepsilon, \theta)$ of the equation $T^u(\varepsilon, t_0) = \theta$ exists and is C^{n+1} in ε for sufficiently small ε. ∎

By Lemma 3.1, $T^u(\varepsilon, t_0)$, $T^s(\varepsilon, t_0)$, $t_0^u(\varepsilon, \theta)$ and $t_0^s(\varepsilon, \theta)$ all have Taylor series expansions to the n^{th}-order in ε near $\varepsilon = 0$. Even though we are unable to find closed forms of those functions, we can still find the coefficients of their Taylor series expansions as demonstrated below. Then, the calculation of $M_k^u(\theta)$ and $M_k^s(\theta)$, for any k with $1 \le k \le n$ is achieved as follows. First, by Lemma 3.2 below, we are able to find the Taylor series expansions of $y_\varepsilon^u(t,t_0)$ and $y_\varepsilon^s(t,t_0)$ to the n^{th}-order in ε near $\varepsilon = 0$. Then, we substitute the Taylor series expansions of $T^u(\varepsilon,t_0)$ and $T^s(\varepsilon,t_0)$ into (20). This enables us to calculate the Taylor series expansions of $\tilde{\Delta}_\varepsilon^u(t_0)$ and $\tilde{\Delta}_\varepsilon^s(t_0)$ to the n^{th}-order as follows,

$$\tilde{\Delta}_\varepsilon^u(t_0) = y_0^u(\tau_0^u) + \tilde{M}_1^u(t_0)\,\varepsilon + \cdots + \tilde{M}_n^u(t_0)\,\varepsilon^n + O\left(\varepsilon^{n+1}\right), \tag{24}$$

$$\tilde{\Delta}_\varepsilon^s(t_0) = y_0^s(\tau_0^s) + \tilde{M}_1^s(t_0)\,\varepsilon + \cdots + \tilde{M}_n^s(t_0)\,\varepsilon^n + O\left(\varepsilon^{n+1}\right). \tag{25}$$

For the final step, we substitute the Taylor series expansions of $t_0^u(\varepsilon,\theta)$ and $t_0^s(\varepsilon,\theta)$ into (24),(25). Then, the Taylor series expansions of $\tilde{M}_1^u(t_0),\dots,\tilde{M}_n^u(t_0)$ and $\tilde{M}_1^s(t_0),\dots,\tilde{M}_n^s(t_0)$ can be computed. Collecting the like terms of ε^k for any k, $1 \le k \le n$ in (24),(25) yields the Taylor series expansions of (23) to the n^{th}-order in ε near $\varepsilon = 0$, which give $M_k^u(\theta)$ and $M_k^s(\theta)$ for any k, $1 \le k \le n$, as desired.

For simplicity we shall use superscript "u, s" to denote the superscripts "u" or "s" in the sequel. Our main result of this section is the following.

THEOREM 3.1. *Let x_0^u, y_0^u, x_0^s, y_0^s, and τ_0^u, τ_0^s be given together with (18) and (19), respectively. Then, the first-order Melnikov function $M_1(\theta)$ is calculated by*

$$M_1(\theta) = -\rho\sqrt{-2G(1)} + \frac{1}{\sqrt{-2G(1)}}\left\{\int_{-\infty}^{\tau_0^u} f_1\left(\tau+\theta-\tau_0^u, x_0^u(\tau), y_0^u(\tau)\right) y_0^u(\tau)\,d\tau \right.$$
$$\left. + \int_{\tau_0^s}^{+\infty} f_1\left(\tau+\theta-\tau_0^s, x_0^s(\tau), y_0^s(\tau)\right) y_0^s(\tau)\,d\tau\right\},$$

where the functions G and f_1 are given by (16) and (4).

The following lemma is derived from the result of Melnikov [16] and is needed in the proof of Theorem 3.1 as well as in the next section.

LEMMA 3.2. *There exists $\tilde{\varepsilon}_0 > 0$ such that, when $0 < |\varepsilon| < \tilde{\varepsilon}_0 < \varepsilon_0 \ll 1$, for any $t_0 \in \mathbb{R}$, the orbits $\gamma_\varepsilon^u : (x_\varepsilon^u(t,t_0), y_\varepsilon^u(t,t_0))$ and γ_ε^s $(x_\varepsilon^s(t,t_0), y_\varepsilon^s(t,t_0))$ described above satisfy the following conditions.*

(i) *$(x_\varepsilon^{u,s}(t,t_0), y_\varepsilon^{u,s}(t,t_0))$ can be expressed as*

$$\begin{aligned} x_\varepsilon^{u,s}(t,t_0) &= x_0^{u,s}(t-t_0) + \sum_{k=1}^{n} x_k^{u,s}(t,t_0)\,\varepsilon^k + O\left(\varepsilon^{n+1}\right), \\ y_\varepsilon^{u,s}(t,t_0) &= y_0^{u,s}(t-t_0) + \sum_{k=1}^{n} y_k^{u,s}(t,t_0)\,\varepsilon^k + O\left(\varepsilon^{n+1}\right) \end{aligned} \tag{26}$$

uniformly with respect to either $t \in (-\infty, t_0]$ for the superscript "u" or $t \in [t_0, +\infty)$ for "s".

(ii) *The initial values $(x_\varepsilon^{u,s}(t_0,t_0), y_\varepsilon^{u,s}(t_0,t_0))$ satisfy*

$$\left(x_\varepsilon^{u,s}(t_0,t_0) - x_0^{u,s}(0)\right)\cdot y_0^{u,s}(0) + \left(y_\varepsilon^{u,s}(t_0,t_0) - y_0^{u,s}(0)\right)\cdot\left(-g\left(x_0^{u,s}(0)\right)\right) = 0.$$

(iii) *There is a constant $M_* > 0$, such that $|x_\varepsilon^u(t,t_0)| + |y_\varepsilon^u(t,t_0)| \le M_*$, for all $t \in (-\infty, t_0]$ and $|x_\varepsilon^s(t,t_0)| + |y_\varepsilon^s(t,t_0)| \le M_*$, for all $t \in [t_0, +\infty)$.*

From now on, if not stated explicitly, we consider all results in the time interval $t \in (-\infty, t_0]$ for the superscript "u" and $t \in [t_0, +\infty)$ for "s".

It is well known (see [12, p. 187]) that $(x_1^{u,s}(t,t_0), y_1^{u,s}(t,t_0))$ satisfy the first variational equation,

$$\begin{aligned} \dot{x}_1^{u,s}(t,t_0) &= y_1^{u,s}(t,t_0), \\ \dot{y}_1^{u,s}(t,t_0) &= -g'\left(x_0^{u,s}(t-t_0)\right) x_1^{u,s}(t,t_0) + f_1\left(t, x_0^{u,s}(t-t_0), y_0^{u,s}(t-t_0)\right). \end{aligned} \tag{27}$$

LEMMA 3.3. *Let*

$$\varphi^{u,s}(t) := \frac{g\left(x_0^{u,s}(t-t_0)\right)}{y_0^{u,s}(t-t_0)}, \qquad \sigma_1^{u,s}(t) := \varphi^{u,s}(t)\, x_1^{u,s}(t,t_0) + y_1^{u,s}(t,t_0). \tag{28}$$

Then,

$$\sigma_1^u\left(t_0+\tau_0^u\right) = \frac{1}{\sqrt{-2G(1)}} \int_{-\infty}^{\tau_0^u} f_1\left(\tau+t_0, x_0^u(\tau), y_0^u(\tau)\right) y_0^u(\tau)\, d\tau,$$
$$\sigma_1^s\left(t_0+\tau_0^s\right) = \frac{1}{\sqrt{-2G(1)}} \int_{\tau_0^s}^{+\infty} f_1\left(\tau+t_0, x_0^s(\tau), y_0^s(\tau)\right) y_0^s(\tau)\, d\tau.$$

PROOF. From (27) and (28), we get

$$\frac{d\sigma_1^{u,s}}{dt} = \varphi^{u,s}(t)\,\sigma_1^{u,s}(t) + f_1\left(t, x_0^{u,s}(t-t_0), y_0^{u,s}(t-t_0)\right). \tag{29}$$

Solving equation (29) yields

$$\begin{aligned}
\sigma_1^{u,s}\left(t_0+\tau_0^{u,s}\right) &= \sigma_1^{u,s}(t)\,\psi^{u,s}(t) - \int_{t_0+\tau_0^{u,s}}^{t} f_1\left(\tau, x_0^{u,s}(\tau-t_0),\ y_0^{u,s}(\tau-t_0)\right)\psi^{u,s}(\tau)\, d\tau \\
&= \sigma_1^{u,s}(t)\,\psi^{u,s}(t) - \int_{\tau_0^{u,s}}^{t-t_0} f_1\left(\tau+t_0, x_0^{u,s}(\tau),\ y_0^{u,s}(\tau)\right)\psi^{u,s}(\tau+t_0)\, d\tau,
\end{aligned} \tag{30}$$

where

$$\psi^{u,s}(t) := \exp\left(-\int_{t_0+\tau_0^{u,s}}^{t} \varphi^{u,s}(\tau)\, d\tau\right) = \frac{y_0^{u,s}(t-t_0)}{y_0^{u,s}(\tau_0^u)}.$$

Following the same arguments as in [12, p. 189], we have

$$\lim_{t\to-\infty} \sigma_1^u(t)\,\psi^u(t) = \lim_{t\to+\infty} \sigma_1^s(t)\,\psi^s(t) = 0,$$

and the improper integrals,

$$\int_{-\infty}^{\tau_0^u} f_1\left(\tau+t_0, x_0^u(\tau),\ y_0^u(\tau)\right) y_0^u(\tau)\, d\tau, \qquad \int_{\tau_0^s}^{+\infty} f_1\left(\tau+t_0, x_0^s(\tau),\ y_0^s(\tau)\right) y_0^s(\tau)\, d\tau,$$

converge. Note that $y_0^u(\tau_0^u) = \sqrt{-2G(1)}$ and $y_0^s(\tau_0^s) = -\sqrt{-2G(1)}$. The proof is completed by letting $t \to +\infty$ for the superscript "s" and $t \to -\infty$ for the superscript "u" in (30). ∎

Now, we are ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. The times $T^u(\varepsilon, t_0)$ and $T^s(\varepsilon, t_0)$ at which the orbits γ_ε^u and γ_ε^s reach the wall $x = 1$, respectively, are determined by the following equations,

$$\begin{aligned}
1 &= x_\varepsilon^{u,s}\left(T^{u,s}(\varepsilon,t_0), t_0\right) \\
&= x_0^{u,s}\left(T^{u,s}(\varepsilon,t_0) - t_0\right) + x_1^{u,s}\left(T^{u,s}(\varepsilon,t_0), t_0\right)\varepsilon + O\left(\varepsilon^2\right).
\end{aligned} \tag{31}$$

Clearly, $T^u(\varepsilon, t_0) < t_0$ and $T^s(\varepsilon, t_0) > t_0$ for sufficiently small ε. Suppose that $T^{u,s}(\varepsilon, t_0)$ have the expansion,

$$T^{u,s}(\varepsilon,t_0) = T_0^{u,s}(t_0) + T_1^{u,s}(t_0)\,\varepsilon + O\left(\varepsilon^2\right). \tag{32}$$

Substituting (32) into (31) and comparing coefficients of ε^0 and ε, we get

$$T_0^{u,s}(t_0) = t_0 + \tau_0^{u,s}, \qquad T_1^{u,s}(t_0) = -\frac{x_1^{u,s}\left(t_0+\tau_0^{u,s}, t_0\right)}{y_0^{u,s}(\tau_0^u)}.$$

Substituting (32) into (20) yields

$$\tilde{\Delta}_{\varepsilon}^{u,s}(t_0) = y_{\varepsilon}^{u,s}\left(T^{u,s}(\varepsilon, t_0), t_0\right) = y_0^{u,s}\left(\tau_0^{u,s}\right) + \tilde{M}_1^{u,s}(t_0)\,\varepsilon + O\left(\varepsilon^2\right), \tag{33}$$

where

$$\tilde{M}_1^{u,s}(t_0) = y_1^{u,s}\left(t_0 + \tau_0^{u,s}, t_0\right) + \dot{y}_0^{u,s}\left(\tau_0^{u,s}\right) T_1^{u,s}(t_0) = \sigma_1^{u,s}\left(t_0 + \tau_0^{u,s}\right). \tag{34}$$

By (34) and Lemma 3.3, we have

$$\tilde{M}_1^{u}(t_0) = \frac{1}{\sqrt{-2G(1)}} \int_{-\infty}^{\tau_0^u} f_1\left(\tau + t_0, x_0^u(\tau), y_0^u(\tau)\right) y_0^u(\tau)\, d\tau, \tag{35}$$

$$\tilde{M}_1^{s}(t_0) = \frac{1}{\sqrt{-2G(1)}} \int_{\tau_0^s}^{+\infty} f_1\left(\tau + t_0, x_0^s(\tau), y_0^s(\tau)\right) y_0^s(\tau)\, d\tau. \tag{36}$$

Let $t_0^{u,s}(\varepsilon,\theta) = \eta_0^{u,s}(\theta) + O(\varepsilon)$. From the identity $\theta = T^{u,s}(\varepsilon, t_0^{u,s}(\varepsilon,\theta))$ and the expansion (32), we obtain $\eta_0^{u,s}(\theta) = \theta - \tau_0^{u,s}$. Substituting $t_0^{u,s}(\varepsilon,\theta) = \eta_0^{u,s}(\theta) + O(\varepsilon)$ into (23) and considering the expansions (11), (12), and (33), we obtain

$$M_1^{u,s}(\theta) = \tilde{M}_1^{u,s}\left(\eta_0^{u,s}(\theta)\right) = \tilde{M}_1^{u,s}\left(\theta - \tau_0^{u,s}\right). \tag{37}$$

Finally, from (15), we have

$$M_1(\theta) = -\rho\sqrt{-2G(1)} + M_1^u(\theta) + M_1^s(\theta). \tag{38}$$

The proof is completed by substituting (35),(36) into (37) then substituting (37) into (38). ∎

4. HIGHER-ORDER MELNIKOV FUNCTIONS

In this section, we give a procedure to compute higher-order Melnikov functions. This is necessary if $M_1(\theta) \equiv 0$, for $\theta \in [0,T]$. We adopt the convention that the sum $\sum_{k=m_1}^{m_2} \alpha_k$ equals 0 if $m_1 > m_2$.

First, we derive equations satisfied by $(x_k^{u,s}(t,t_0),\ y_k^{u,s}(t,t_0))$, for $1 \le k \le n$. It is clear by Lemma 3.2 that $(x_\varepsilon^{u,s}(t,t_0),\ y_\varepsilon^{u,s}(t,t_0))$ are uniquely determined and, for any k with $1 \le k \le n$, we have

$$\begin{aligned} x_k^{u,s}(t_0,t_0)\, y_0^{u,s}(0) - y_k^{u,s}(t_0,t_0)\, g\left(x_0^{u,s}(0)\right) = 0, \\ \left|x_k^{u,s}(t,t_0)\right| + \left|y_k^{u,s}(t,t_0)\right| \le M_*. \end{aligned} \tag{39}$$

Let

$$A_{mk}^{u,s} := \begin{cases} x_k^{u,s}(t,t_0), & \text{if } m = 1 \text{ and } 1 \le k \le n, \\ \sum\limits_{i=m-1}^{k-1} A_{m-1,i}^{u,s} x_{k-i}^{u,s}(t,t_0), & \text{if } 2 \le m \le k \le n, \end{cases}$$

$$B_{mk}^{u,s} := \begin{cases} y_k^{u,s}(t,t_0), & \text{if } m = 1 \text{ and } 1 \le k \le n, \\ \sum\limits_{i=m-1}^{k-1} B_{m-1,i}^{u,s} y_{k-i}^{u,s}(t,t_0), & \text{if } 2 \le m \le k \le n, \end{cases}$$

$$C_{mkl}^{u,s} := \begin{cases} A_{ml}^{u,s}, & \text{if } k = 0 \text{ and } 1 \le m \le l \le n, \\ B_{kl}^{u,s}, & \text{if } m = 0 \text{ and } 1 \le k \le l \le n, \\ \sum\limits_{i=m}^{l-k} A_{mi}^{u,s} B_{k,l-i}^{u,s}, & \text{if } m \ge 1, \quad k \ge 1 \text{ and } m + k \le l \le n, \end{cases}$$

and for $m \ge 0$, $k \ge 0$, $1 \le m + k \le l \le n$, and $1 \le j \le n$, let

$$\tilde{C}_{mklj}^{u,s} := \frac{C_{mkl}^{u,s}}{m!k!} \frac{\partial^{m+k}}{\partial x^m \partial y^k} f_j\left(t, x_0^{u,s}(t - t_0),\ y_0^{u,s}(t - t_0)\right).$$

Then, for $1 \le m \le n$, we have

$$[x_\varepsilon^{u,s}(t,t_0) - x_0^{u,s}(t-t_0)]^m = \sum_{k=m}^{n} A_{mk}^{u,s}\varepsilon^k + O\left(\varepsilon^{n+1}\right),$$

$$[y_\varepsilon^{u,s}(t,t_0) - y_0^{u,s}(t-t_0)]^m = \sum_{k=m}^{n} B_{mk}^{u,s}\varepsilon^k + O\left(\varepsilon^{n+1}\right),$$

and for $0 \le m,\ k \le n$, with $1 \le m+k \le n$,

$$[x_\varepsilon^{u,s}(t,t_0) - x_0^{u,s}(t-t_0)]^m\,[y_\varepsilon^{u,s}(t,t_0) - y_0^{u,s}(t-t_0)]^k = \sum_{l=m+k}^{n} C_{mkl}^{u,s}\varepsilon^l + O\left(\varepsilon^{n+1}\right).$$

It follows from the expansion (26) that

$$\begin{aligned} g(x_\varepsilon^{u,s}(t,t_0)) &= g(x_0^{u,s}(t-t_0)) \\ &\quad + \sum_{k=1}^{n}\left\{\sum_{m=1}^{k}\frac{A_{mk}^{u,s}}{m!}g^{(m)}(x_0^{u,s}(t-t_0))\right\}\varepsilon^k + O(\varepsilon^{n+1}), \end{aligned} \tag{40}$$

$$\varepsilon f(t, x_\varepsilon^{u,s}(t,t_0), y_\varepsilon^{u,s}(t,t_0)\varepsilon) = \sum_{k=1}^{n} F_k^{u,s}(t,t_0)\varepsilon^k + O(\varepsilon^{n+1}), \tag{41}$$

where

$$F_k^{u,s}(t,t_0) := \begin{cases} f_1\left(t, x_0^{u,s}(t-t_0),\ y_0^{u,s}(t-t_0)\right), & \text{if } k=1, \\ f_k\left(t, x_0^{u,s}(t-t_0),\ y_0^{u,s}(t-t_0)\right) + \sum\limits_{l=1}^{k-1}\sum\limits_{i=0}^{k-l}\tilde{C}_{i,k-l-i,k-l,l}^{u,s}, & \text{if } 2 \le k \le n. \end{cases}$$

Obviously, for any k with $2 \le k \le n$, $F_k^{u,s}(t,t_0)$ only depends on $(x_0^{u,s}(t-t_0),\ y_0^{u,s}(t-t_0))$, $(x_1^{u,s}(t,t_0),\ y_1^{u,s}(t,t_0)), \ldots, (x_{k-1}^{u,s}(t,t_0),\ y_{k-1}^{u,s}(t,t_0))$. Substituting (26), (40), and (41) into (9) and then comparing the coefficients of ε^k for each k, $1 \le k \le n$, we obtain the following linear system,

$$\begin{aligned} &\dot{x}_k^{u,s}(t,t_0) = y_k^{u,s}(t,t_0), \\ &\dot{y}_k^{u,s}(t,t_0) = -g'\left(x_0^{u,s}(t-t_0)\right)x_k^{u,s}(t,t_0) + F_k^{u,s}(t,t_0). \end{aligned} \tag{42}$$

The corresponding homogenous linear system of (42) has a fundamental matrix solution,

$$\begin{pmatrix} y_0^{u,s}(t-t_0) & y_0^{u,s}(t-t_0)\int_{t_0}^{t}\left(y_0^{u,s}(\tau-t_0)\right)^{-2}d\tau \\ \dot{y}_0^{u,s}(t-t_0) & \dot{y}_0^{u,s}(t-t_0)\int_{t_0}^{t}\left(y_0^{u,s}(\tau-t_0)\right)^{-2}d\tau + \dfrac{1}{y_0^{u,s}(t-t_0)} \end{pmatrix}.$$

Hence, for any k with $1 \le k \le n$, $(x_k^{u,s}(t,t_0), y_k^{u,s}(t,t_0))$ can be uniquely solved from the variational equations (42) with conditions (39) recursively by using the variation-of-constants formula.

Extend (31) and (32) to higher-order Taylor series expansions as following,

$$1 = x_0^{u,s}\left(T^{u,s}(\varepsilon,t_0) - t_0\right) + \sum_{k=1}^{n} x_k^{u,s}\left(T^{u,s}(\varepsilon,t_0), t_0\right)\varepsilon^k + O\left(\varepsilon^{n+1}\right), \tag{43}$$

$$T^{u,s}(\varepsilon,t_0) = T_0^{u,s}(t_0) + \sum_{k=1}^{n} T_k^{u,s}(t_0)\varepsilon^k + O\left(\varepsilon^{n+1}\right). \tag{44}$$

Substituting (44) into (43) and comparing coefficients of ε^k for each k, we have $T_0^{u,s}(t_0) = t_0 + \tau_0^{u,s}$ and for $1 \le k \le n$,

$$\begin{aligned} T_k^{u,s}(t_0) = -\frac{1}{y_0^{u,s}(\tau_0^{u,s})}&\left\{x_k^{u,s}(t_0+\tau_0^{u,s}, t_0) + \sum_{m=2}^{k}\frac{D_{mk}^{u,s}}{m!}\left.\frac{d^m x_0^{u,s}}{dt^m}\right|_{t=\tau_0^{u,s}}\right. \\ &\left. + \sum_{j=1}^{k-1}\sum_{m=1}^{k-j}\frac{D_{m,k-j}^{u,s}}{m!}\left.\frac{d^m x_j^{u,s}}{dt^m}\right|_{t=t_0+\tau_0^{u,s}}\right\}, \end{aligned} \tag{45}$$

where

$$D_{mk}^{u,s} := \begin{cases} T_k^{u,s}(t_0), & \text{if } m = 1 \text{ and } 1 \le k \le n, \\ \sum\limits_{i=m-1}^{k-1} D_{m-1,i}^{u,s} T_{k-i}^{u,s}(t_0), & \text{if } 2 \le m \le k \le n. \end{cases}$$

From (45), we can compute the coefficients $T_1^{u,s}(t_0), \dots, T_n^{u,s}(t_0)$ in the expansion (44) recursively.

Substituting (44) into (20) yields

$$\tilde{\Delta}_\varepsilon^{u,s}(t_0) = y_\varepsilon^{u,s}\left(T^{u,s}(\varepsilon, t_0), t_0\right) = y_0^{u,s}\left(\tau_0^{u,s}\right) + \sum_{k=1}^{n} \tilde{M}_k^{u,s}(t_0)\, \varepsilon^k + O\left(\varepsilon^{n+1}\right), \tag{46}$$

where for $1 \le k \le n$,

$$\begin{aligned} \tilde{M}_k^{u,s}(t_0) = {} & y_k^{u,s}\left(t_0 + \tau_0^{u,s}, t_0\right) + \sum_{m=1}^{k} \frac{D_{mk}^{u,s}}{m!} \left.\frac{d^m y_0^{u,s}}{dt^m}\right|_{t=\tau_0^{u,s}} \\ & + \sum_{j=1}^{k-1} \sum_{m=1}^{k-j} \frac{D_{m,k-j}^{u,s}}{m!} \left.\frac{d^m y_j^{u,s}}{dt^m}\right|_{t=t_0+\tau_0^{u,s}}. \end{aligned} \tag{47}$$

Similar to Lemma 3.3, we have the following lemma.

Lemma 4.1. *Let $\sigma_k^{u,s}(t) := \varphi^{u,s}(t) x_k^{u,s}(t, t_0) + y_k^{u,s}(t, t_0)$, where $2 \le k \le n$. Then,*

$$\sigma_k^u\left(t_0 + \tau_0^u\right) = \frac{1}{\sqrt{-2G(1)}} \int_{-\infty}^{\tau_0^u} F_k^u\left(\tau + t_0, t_0\right) y_0^u(\tau)\, d\tau,$$

$$\sigma_k^s\left(t_0 + \tau_0^s\right) = \frac{1}{\sqrt{-2G(1)}} \int_{\tau_0^s}^{+\infty} F_k^s\left(\tau + t_0, t_0\right) y_0^s(\tau)\, d\tau.$$

From (45) and (47), we see that

$$\tilde{M}_k^{u,s}(t_0) = \sigma_k^{u,s}\left(t_0 + \tau_0^{u,s}\right) + \xi_k^{u,s}(t_0), \tag{48}$$

where

$$\begin{aligned} \xi_k^{u,s}(t_0) = {} & \sum_{m=2}^{k} \frac{D_{mk}^{u,s}}{m!} \left.\left(\frac{d^m y_0^{u,s}}{dt^m} + \varphi^{u,s}\left(t_0 + \tau_0^{u,s}\right) \frac{d^m x_0^{u,s}}{dt^m}\right)\right|_{t=\tau_0^{u,s}} \\ & + \sum_{j=1}^{k-1} \sum_{m=1}^{k-j} \frac{D_{m,k-j}^{u,s}}{m!} \left.\left(\frac{d^m y_j^{u,s}}{dt^m} + \varphi^{u,s}\left(t_0 + \tau_0^{u,s}\right) \frac{d^m x_j^{u,s}}{dt^m}\right)\right|_{t=t_0+\tau_0^{u,s}}. \end{aligned}$$

By (48) and Lemma 4 1, we have

$$\tilde{M}_k^u(t_0) = \frac{1}{\sqrt{-2G(1)}} \int_{-\infty}^{\tau_0^u} F_k^u\left(\tau + t_0, t_0\right) y_0^u(\tau)\, d\tau + \xi_k^u(t_0), \tag{49}$$

$$\tilde{M}_k^s(t_0) = \frac{1}{\sqrt{-2G(1)}} \int_{\tau_0^s}^{+\infty} F_k^s\left(\tau + t_0, t_0\right) y_0^s(\tau)\, d\tau + \xi_k^s(t_0). \tag{50}$$

Since $F_k^{u,s}(t, t_0)$ and $\xi_k^{u,s}(t_0)$ only depend on

$$\left(x_0^{u,s}(t - t_0),\ y_0^{u,s}(t - t_0)\right), \left(x_1^{u,s}(t, t_0),\ y_1^{u,s}(t, t_0)\right), \ \dots, \left(x_{k-1}^{u,s}(t, t_0),\ y_{k-1}^{u,s}(t, t_0)\right),$$

for any k with $1 \le k \le n$, in the computation of $\tilde{M}_k^{u,s}(t_0)$, there is no need to calculate $x_k^{u,s}(t, t_0)$ and $y_k^{u,s}(t, t_0)$.

Let $t_0^{u,s}(\varepsilon,\theta)$ have the following expansion,

$$t_0^{u,s}(\varepsilon,\theta) = \eta_0^{u,s}(\theta) + \sum_{k=1}^{n} \eta_k^{u,s}(\theta)\,\varepsilon^k + O\left(\varepsilon^{n+1}\right). \tag{51}$$

From the identity $\theta = T^{u,s}(\varepsilon, t_0^{u,s}(\varepsilon,\theta))$ and the expansions (44) and (51), we obtain

$$\eta_k^{u,s}(\theta) = \begin{cases} \theta - \tau_0^{u,s}, & \text{if } k = 0, \\ -\displaystyle\sum_{j=1}^{k-1}\sum_{m=1}^{k-j} \frac{\Theta_{m,k-j}^{u,s}}{m!} \left.\frac{d^m T_j^{u,s}}{dt_0^m}\right|_{t_0=\eta_0^{u,s}(\theta)} - T_k^{u,s}\left(\eta_0^{u,s}(\theta)\right), & \text{if } 1 \le k \le n, \end{cases}$$

where

$$\Theta_{mk}^{u,s} := \begin{cases} \eta_k^{u,s}(\theta), & \text{if } m = 1 \text{ and } 1 \le k \le n, \\ \displaystyle\sum_{i=m-1}^{k-1} \Theta_{m-1,i}^{u,s}\,\eta_{k-i}^{u,s}(\theta), & \text{if } 2 \le m \le k \le n. \end{cases}$$

Thus, the coefficients $\eta_0^{u,s}(\theta),\ldots,\eta_n^{u,s}(\theta)$ in the expansion (51) can be computed recursively.

Finally, substituting (51) into (23) and considering the expansions (11), (12), and (46), we obtain

$$M_k^{u,s}(\theta) = \tilde{M}_k^{u,s}\left(\eta_0^{u,s}(\theta)\right) + \sum_{j=1}^{k-1}\sum_{m=1}^{k-j} \frac{\Theta_{m,k-j}^{u,s}}{m!} \left.\frac{d^m \tilde{M}_j^{u,s}}{dt_0^m}\right|_{t_0=\theta-\tau_0^{u,s}}, \tag{52}$$

for $1 \le k \le n$.

Thus, we have obtained a systematic procedure to compute $M_k^u(\theta)$ and $M_k^s(\theta)$ for each integer k, $1 \le k \le n$, and for any $\theta \in [0,T]$. Hence the Melnikov functions $M_k(\theta)$ can be computed. The procedure for the computation of $M_k(\theta)$, which is summarized below, is algorithmic in spirit, thus, computer algebra or numerical routines can be easily developed.

1. Given $(x_0^{u,s}(t),\ y_0^{u,s}(t))$, solve the linear differential system (42) with the conditions in (39) for $(x_j^u(t,t_0),\ y_j^u(t,t_0))$ $(1 \le j \le k-1)$ in the interval $(-\infty, t_0]$ of time t and $(x_j^s(t,t_0),\ y_j^s(t,t_0))$ $(1 \le j \le k-1)$ in the interval $[t_0,+\infty)$ recursively.
2. Find $\tau_0^{u,s}$, the time at which the orbits $(x_0^{u,s}(t), y_0^{u,s}(t))$ reach the wall $x = 1$ by solving the algebraic equations $x_0^{u,s}(\tau_0^{u,s}) = 1$. Then, compute $T_1^{u,s}(t_0),\ \ldots,\ T_{k-1}^{u,s}(t_0)$ recursively using (45).
3. For any $t_0 \in \mathbb{R}$, compute $\tilde{M}_1^{u,s}(t_0),\ldots,\tilde{M}_k^{u,s}(t_0)$ using (49),(50).
4. For any $\theta \in [0,T]$, compute $M_1^{u,s}(\theta),\ldots,M_k^{u,s}(\theta)$ using (52). Then, compute $M_1(\theta),\ldots,M_k(\theta)$ using (15).

The formula for $M_1(\theta)$ is given in Theorem 3.1. With this procedure, we can easily obtain an explicit formula for the second-order Melnikov function $M_2(\theta)$ for $\theta \in [0,T]$,

$$\begin{aligned} M_2(\theta) = -\rho^2\sqrt{-2G(1)} + \frac{1}{\sqrt{-2G(1)}} \Bigg\{ &\rho \int_{\tau_0^s}^{+\infty} f_1\left(\tau+\theta-\tau_0^s, x_0^s(\tau), y_0^s(\tau)\right) y_0^s(\tau)\,d\tau \\ &+ \int_{-\infty}^{\tau_0^u} F_2^u\left(\tau+\theta-\tau_0^u, \theta-\tau_0^u\right) y_0^u(\tau)\,d\tau + \int_{\tau_0^s}^{+\infty} F_2^s\left(\tau+\theta-\tau_0^s, \theta-\tau_0^s\right) y_0^s(\tau)\,d\tau \\ &-T_1^u\left(\theta-\tau_0^u\right) \int_{-\infty}^{\tau_0^u} f_{1t}\left(\tau+\theta-\tau_0^u, x_0^u(\tau), y_0^u(\tau)\right) y_0^u(\tau)\,d\tau \\ &-T_1^s\left(\theta-\tau_0^s\right) \int_{\tau_0^s}^{+\infty} f_{1t}\left(\tau+\theta-\tau_0^s, x_0^s(\tau), y_0^s(\tau)\right) y_0^s(\tau)\,d\tau \Bigg\} \\ &+\xi_2^u\left(\theta-\tau_0^u\right) + \xi_2^s\left(\theta-\tau_0^s\right), \end{aligned} \tag{53}$$

where

$$\xi_2^{u,s}(t_0) = \frac{1}{2} T_1^{u,s}(t_0) \left\{ \varphi^{u,s}(t_0 + \tau_0^u) \left[y_1^{u,s}(t_0 + \tau_0^{u,s}, t_0) + \tilde{M}_1^{u,s}(t_0) \right] \right.$$
$$\left. + \dot{y}_1^{u,s}(t_0 + \tau_0^{u,s}, t_0) + f_1(t_0 + \tau_0^{u,s}, x_0^{u,s}(\tau_0^s),\ y_0^{u,s}(\tau_0^{u,s})) \right\},$$
$$F_2^{u,s}(t, t_0) = \frac{1}{2} \dot{g}(x_0^{u,s}(t - t_0))(x_1^{u,s}(t, t_0))^2 + f_{1x}(t, x_0^{u,s}(t - t_0),\ y_0^{u,s}(t - t_0))\, x_1^{u,s}(t, t_0)$$
$$+ f_{1y}(t, x_0^{u,s}(t - t_0),\ y_0^{u,s}(t - t_0))\, y_1^{u,s}(t, t_0)$$
$$+ f_2(t, x_0^{u,s}(t - t_0),\ y_0^{u,s}(t - t_0)).$$

Here, f_{1t}, f_{1x}, f_{1y} denote, respectively, the partial derivatives of the function $f_1(t, x, y)$ with respect to t, x, y, and f_1, f_2 are given by (4).

5. APPLICATIONS

The piecewise linear impact system,

$$x + 2\varepsilon\alpha\dot{x} - x = \varepsilon\beta\cos\omega t, \qquad \text{as } |x| < 1,$$
$$\dot{x} \mapsto -(1 - \varepsilon\rho)\,\dot{x}, \qquad \text{as } |x| = 1,$$

where $\varepsilon\alpha$ is a linear damping coefficient and $\varepsilon\beta$ is the forcing amplitude, was discussed by Shaw and Rand in [9]. Take

$$(x_0^u(t),\ y_0^u(t)) = (e^{t+T}, e^{t+T}), \qquad t \in (-\infty, 0],$$
$$(x_0^s(t),\ y_0^s(t)) = (e^{-t+T}, -e^{-t+T}), \qquad t \in [0, +\infty),$$

where $T = 2\pi/\omega$. Then, $\tau_0^u = -T$ and $\tau_0^s = T$. By Theorem 3.1 and (53),

$$M_1(\theta) = -\rho - 2\alpha + \frac{2\omega\beta}{1 + \omega^2} \sin\omega\theta,$$
$$M_2(\theta) = -\rho(\alpha + \rho) + \frac{\beta}{1 + \omega^2} \left\{ \rho\omega\sin\omega\theta - \left[\rho + \frac{2\alpha(1 - \omega^2)}{1 + \omega^2} \right] \cos\omega\theta \right\},$$

where $\theta \in [0, 2\pi/\omega]$. It is easy to check that our results agree with those in [9].

Here, we compute $M_2(\theta)$ for this system to illustrate our method of calculation, although it is indeed unnecessary because $M_1(\theta) \not\equiv 0$ for $\theta \in [0, 2\pi/\omega]$.

Since our method does not depend upon the closed form solutions of the perturbed system between the impact walls, it can be easily applied to nonlinear impact systems. Consider an nonlinear impact system whose free flight motion between the walls is governed by the Duffing equation,

$$\ddot{x} + \varepsilon\delta\dot{x} - x + x^3 = \varepsilon\gamma\cos\omega t, \qquad \text{for } |x| < 1,$$

and the impact rule is given by (3). The system is equivalent to

$$\dot{x} = y, \quad \dot{y} = x - x^3 + \varepsilon(-\delta y + \gamma\cos\omega t), \qquad \text{as } |x| < 1, \tag{54}$$
$$y \mapsto -(1 - \varepsilon\rho)\, y, \qquad \text{as } |x| = 1. \tag{55}$$

When $\varepsilon = 0$, the phase portrait of (54),(55) is topologically equivalent to that shown in Figure 2. Here, $T = 2\pi/\omega$. Take

$$(x_0^u(t), y_0^u(t)) = \left(\sqrt{2}\ \mathrm{sech}\ t, -\sqrt{2}\ \mathrm{sech}\ t\tanh t\right), \qquad t \in (-\infty, 0],$$
$$(x_0^s(t), y_0^s(t)) = \left(\sqrt{2}\ \mathrm{sech}\ t, -\sqrt{2}\ \mathrm{sech}\ t\tanh t\right), \qquad t \in [0, +\infty).$$

Let $\kappa = \ln(\sqrt{2}+1)$. Then, $\tau_0^u = -\kappa$ and $\tau_0^s = \kappa$. By Theorem 3.1,

$$M_1(\theta) = -\frac{\rho}{\sqrt{2}} - \frac{2\delta}{3}\left(2\sqrt{2}-1\right) - 2\gamma\cos(\omega\kappa)\left\{\int_{-\infty}^{-\kappa}\phi_1(\tau)\,d\tau + \int_{\kappa}^{+\infty}\phi_1(\tau)\,d\tau\right\}$$
$$+ 2\gamma\sin(\omega\kappa)\left\{\int_{-\infty}^{-\kappa}\phi_2(\tau)\,d\tau - \int_{\kappa}^{+\infty}\phi_2(\tau)\,d\tau\right\},$$

where $\phi_1(t) :=$ sech $t\tanh t\cos\omega(t+\theta)$ and $\phi_2(t) :=$ sech $t\tanh t\sin\omega(t+\theta)$. Further calculation yields

$$M_1(\theta) = -\frac{\rho}{\sqrt{2}} - \frac{2\delta}{3}\left(2\sqrt{2}-1\right) + 4\gamma\omega\left\{\frac{\pi}{2}\cos(\omega\kappa)\ \text{sech}\left(\frac{\pi\omega}{2}\right) - \chi(\omega)\right\}\sin(\omega\theta), \tag{56}$$

for $\theta \in [0, 2\pi/\omega]$, where $\kappa = \ln(\sqrt{2}+1)$ and

$$\chi(\omega) = \cos(\omega\kappa)\int_0^{\kappa}\text{sech}\ \tau\cos(\omega\tau)\,d\tau - \sin(\omega\kappa)\int_{\kappa}^{+\infty}\text{sech}\ \tau\sin(\omega\tau)\,d\tau.$$

For any given ω, $\chi(\omega)$ can be easily evaluated numerically to the desired accuracy.

Define

$$\gamma_\infty(\delta,\rho,\omega) := \left|\frac{3\rho + 2\delta\left(4-\sqrt{2}\right)}{12\sqrt{2}\omega\left\{(\pi/2)\cos(\omega\kappa)\ \text{sech}\ (\pi\omega/2) - \chi(\omega)\right\}}\right|.$$

From (56) and by Theorem 2.1, we conclude that if $|\gamma| > \gamma_\infty$ then, for sufficiently small ε, W_ε^s and W_ε^u intersect transversally in Σ and that if $|\gamma| < \gamma_\infty$, then W_ε^s and W_ε^u do not intersect in Σ. Moreover, since $M_1(\theta)$ has quadratic zeros when $|\gamma| = \gamma_\infty$, quadratic tangency occurs between W_ε^s and W_ε^u in this case.

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